

Percolation, Bose-Einstein condensation, and string proliferation

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The close analogy between cluster percolation and string proliferation in the context of critical phenomena is studied. Like clusters in percolation theory, closed strings, which can be either finite-temperature worldlines or topological line defects, are described by a distribution parametrized by only two exponents. On approaching the critical point, the string tension vanishes and the loops proliferate, thereby signalling the onset of Bose-Einstein condensation (in the case of worldlines) or the disordering of the ordered state (in the case of vortices). The ideal Bose gas with modified energy spectrum is used as a stepping stone to derive general expressions for the critical exponents in terms of the two exponents parametrizing the loop distribution near criticality.

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I. INTRODUCTION

The quest for a geometrical description of phase transitions has a long history going back to ideas first put forward by Onsager [1] in the context of the λ transition in liquid ^4He . The relevant geometrical objects in this transition are topological line defects. The description envisaged by Onsager is one entirely in terms of these one-dimensional objects, with their geometrical properties such as fractal dimension and configurational entropy. The phase transition is characterized in this picture by a fundamental change in the typical string size. Whereas in the superfluid phase only finite strings are present, at the critical point infinite strings appear—similar to the sudden appearance of a percolating cluster in percolation phenomena at criticality.

The analogy between percolation of clusters and proliferation of strings has been noted by various authors [2–5]. As we will see, it derives from a similarity in the cluster and string distribution. Both have the same form containing two factors, one related to the entropy of a given cluster or string configuration and the other related to the Boltzmann weight assigned to the configuration. Close to criticality, each of these factors is parametrized by a single exponent. One specifies the algebraic behavior of the distribution at criticality, while the other describes how the Boltzmann factor tends to unity upon approaching the critical point. Physically, the unity of the Boltzmann factor implies that clusters or strings can grow, and thus, gain configurational entropy without energy cost. When the Boltzmann factor is not unity, the clusters and strings are exponentially suppressed. In the context of strings, the transition between a phase consisting of finite strings only and one having infinite strings, is called a Hagedorn transition. We will refer to the appearance of infinite strings as proliferation.

A central position in our arguments is taken by an ideal Bose gas with modified energy spectrum. The reason is that although noninteracting, the model has nontrivial critical exponents, which are known exactly, while at the same time it can also be mapped onto a loop gas of worldlines in an exact way. The worldlines form closed loops, because in the absence of external fields and sources they cannot terminate in the system. The map onto the loop gas allows us to connect

the critical exponents of the phase transition to the two exponents parametrizing the worldline loop distribution near criticality. Using general scaling relations, these results can then be generalized to interacting loop gases representing statistical models. Each universality class is defined by a loop distribution with specific values for the two exponents from which all the critical exponents follow.

To close the circle, the resulting loop gas description of critical phenomena can also be applied to phase transitions involving (interacting) vortex loops—like the λ transition in liquid ^4He . As for worldlines, vortices cannot terminate in the system, and therefore, form closed loops too. When the word string is used in the following, the two different one-dimensional objects worldlines and vortices should be kept in mind.

As an aside, we believe that the theory of vertex loop gases discussed here has also a bearing on homogeneous superfluid turbulence.

The paper is organized as follows. In the next section those essentials of percolation theory are recalled, which later on in the paper become important to establish the connection with the proliferation of strings. In Sec. III, Bose-Einstein condensation in an ideal Bose gas with a modified energy spectrum is studied from the perspective of worldlines. In Sec. IV, numerical work on random string networks is discussed in relation to uncorrelated percolation, followed by a discussion of correlated percolation in Sec. V. In Sec. VI, thermal phase transitions are formulated in terms of proliferating worldlines. In Sec. VII, a similar formulation is discussed for phase transitions involving proliferating vortices. The paper ends with conclusions in Sec. VIII.

II. REVIEW OF SITE PERCOLATION

In this section, we briefly recall some basic aspects of percolation theory [6] that are important for our purposes. Consider a lattice, with each lattice site being randomly occupied with a probability p , say, independent of its neighbors. Islands of next-neighboring occupied sites form clusters. The properties of these clusters as a function of p form the subject of percolation theory—or more precisely, of *uncorrelated* percolation theory because the occupations of dif-

ferent lattice sites are uncorrelated. As p increases, clusters become bigger, and at some critical value p_c , a cluster spanning the entire lattice first appears. Near this percolation threshold, various quantities show power-law behavior. As for thermal critical phenomena, this behavior can be characterized by a set of critical exponents of which two are independent.

Of interest to us is the so-called percolation strength $P(p) \sim (p - p_c)^{\beta_{\text{per}}}$ defined for $p > p_c$ as the probability that a randomly chosen site belongs to the percolating cluster, the average cluster size $S(p) \sim |p_c - p|^{-\gamma_{\text{per}}}$, and the correlation length $\xi(p) \sim |p_c - p|^{-\nu}$. We give some quantities a subscript ‘‘per’’ to avoid confusion later on, where similar, but different quantities appear. The percolation strength is analogous to the magnetization in spin models, while the average cluster size corresponds to the magnetic susceptibility.

The number density of clusters having size s , i.e., with s occupied sites, is assumed to be distributed according to

$$l_s(p) \propto s^{-\tau} \exp(-cs), \quad (1)$$

where the coefficient c vanishes with an exponent $1/\sigma$ when the percolation threshold p_c is approached from below, $c \propto (p_c - p)^{1/\sigma}$. At criticality, the cluster distribution becomes $l_s(p_c) \propto s^{-\tau}$ and falls off algebraically. This factor measures the configurational entropy of clusters, while the exponential is a Boltzmann factor. Together with the so-called Fisher exponent τ , the exponent σ determines the critical exponents, such as β_{per} , γ_{per} , and ν through scaling relations.

In terms of the cluster distribution l_s , the average cluster size is given by

$$S(p) = \frac{\sum_s s^2 l_s(p)}{\sum_s s l_s(p)}, \quad (2)$$

where only finite clusters are included in the sum. To understand that S thus defined indeed is a measure for the average cluster size, note that the combination $s l_s(p)$ in the denominator is the probability that a randomly chosen site belongs to a cluster of size s . When summed over s , this gives the probability that the randomly chosen site belongs to a cluster of arbitrary (finite) size. The ratio $s l_s(p) / \sum_s s l_s(p)$ is therefore the probability that the cluster to which the randomly chosen site belongs is of size s . When this ratio is multiplied with the cluster size and summed over s , one obtains indeed a measure of the average cluster size. With the explicit expression for S , one easily finds that

$$-\gamma_{\text{per}} = \frac{\tau - 3}{\sigma}. \quad (3)$$

For later reference, recall that the susceptibility also equals the integral over the correlation function $G_{\text{per}}(x)$ of the system under consideration. Because of Eq. (2), one finds for percolation

$$\sum_x G_{\text{per}}(x) \propto \sum_s s^2 l_s(p), \quad (4)$$

where the sum \sum_x is over all lattice sites. Physically, $G_{\text{per}}(x)$ is the probability of finding at a distance x from the origin an occupied site belonging to the same cluster as the origin does. At criticality, it has in d space dimensions the algebraic behavior

$$G_{\text{per}}(x) \sim \frac{1}{x^{d-2+\eta_{\text{per}}}}, \quad (5)$$

with η_{per} a critical exponent.

To obtain a similar expression for the percolation strength as given in Eq. (2) for S , one uses the identity

$$p = P(p) + \sum_s s l_s(p), \quad (6)$$

stating that an occupied site either belongs to the percolating cluster or to a finite one. Hence, for $p > p_c$,

$$P(p) \sim - \sum_s s l_s(p), \quad (7)$$

and

$$\beta_{\text{per}} = \frac{\tau - 2}{\sigma}. \quad (8)$$

Below the percolation threshold, $P(p) = 0$, so that the identity becomes $p = \sum_s s l_s(p)$.

Given the analogy with spin systems, it should be possible to define a partition function Z from which the percolation strength (magnetization) and the average cluster size (susceptibility) can be obtained by differentiation. Indeed, with the definition

$$\ln(Z) \propto \sum_s l_s, \quad (9)$$

one finds

$$P = \frac{\partial \ln(Z)}{\partial c}, \quad S = \frac{\partial^2 \ln(Z)}{\partial c^2}, \quad (10)$$

as required by the analogy. Physically, $\ln(Z)$ denotes the total number density of finite clusters.

The two critical exponents γ_{per} and β_{per} can be related to the critical exponent α characterizing the algebraic behavior of $\ln(Z) \sim |p_c - p|^{2-\alpha}$ close to the percolation threshold as follows:

$$2 - \alpha = \frac{\tau - 1}{\sigma} = 2\beta_{\text{per}} + \gamma_{\text{per}}. \quad (11)$$

On dimensional grounds, one expects that near the percolation threshold $\ln(Z) \sim \xi^{-d} \sim |p - p_c|^{d\nu}$, so that one arrives at the hyperscaling relation

$$d\nu = 2 - \alpha, \quad (12)$$

involving the dimensionality d of the lattice. In the context of thermal critical phenomena, relations (11) and (12) are known as Rushbrook's and Josephson's scaling law, respectively. A hyperscaling relation in general breaks down at the upper critical dimension. Beyond the upper critical dimension d_u , the critical exponents are locked in their values at $d = d_u$. For percolation, as we will see below, the upper critical dimension is $d_u = 6$.

Another scaling relation between the various critical exponents is obtained from the expressions (4) and (5) involving the correlation function. Using ξ as lower cutoff in the integral, one finds

$$\gamma_{\text{per}} = \nu(2 - \eta_{\text{per}}), \quad (13)$$

which in the context of thermal-phase transitions is known as Fisher's scaling law.

The correlation length exponent ν is known to have a geometric meaning, being related to the Hausdorff, or fractal dimension D at criticality. The latter can be defined through the average squared distance between the sites of a cluster,

$$R_s^2 = \frac{1}{s} \sum_{i=1}^s (\mathbf{x}_i - \bar{\mathbf{x}})^2 = \frac{1}{2s^2} \sum_{i,j=1}^s (\mathbf{x}_i - \mathbf{x}_j)^2, \quad (14)$$

with R_s the so-called radius of gyration, and $\bar{\mathbf{x}} = (1/s) \sum_{i=1}^s \mathbf{x}_i$ the center of mass of the cluster with its sites at \mathbf{x}_i , as

$$R_s \sim s^{1/D} \quad (15)$$

for large enough clusters. A standard definition of the correlation length in terms of the correlation function $G(x)$, with $x = |\mathbf{x}|$, reads

$$\xi^2 \propto \frac{\sum_x x^2 G(x)}{\sum_x G(x)}. \quad (16)$$

It measures the average distance between two sites in the same cluster. Because of the observation (4), this definition can be transcribed in terms of the cluster distribution as

$$\xi^2 \propto 2 \frac{\sum_s R_s^2 s^2 l_s(p)}{\sum_s s^2 l(p)}. \quad (17)$$

With Eq. (15), the relation between the correlation length and the fractal dimension is then easily found to be

$$\frac{1}{D} = \nu\sigma. \quad (18)$$

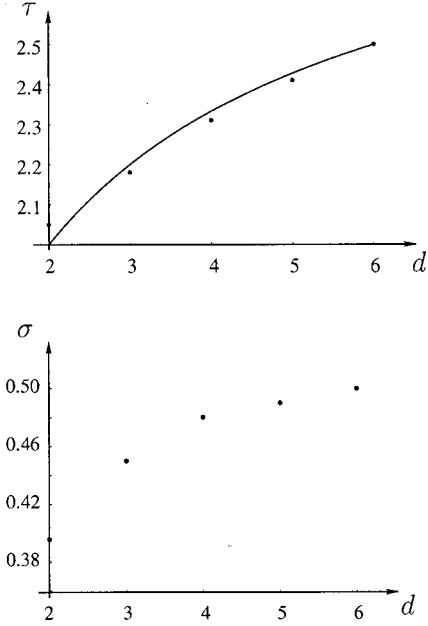


FIG. 1. The Fisher exponent τ and the exponent σ parametrizing the cluster distribution of uncorrelated site percolation as function of the dimensionality d of the lattice. The points represent results [6] obtained by lattice simulations ($d=3,4,5$) and analytic methods ($d=2,6$), while the line in the top panel is based on de Gennes's estimate (20) that can be given for any dimension $2 \leq d \leq 6$, not just integer values.

Combined with this relation, the hyperscaling relation (12) leads to the well-known expression for the Fisher exponent τ in terms of the fractal dimension,

$$\tau = \frac{d}{D} + 1. \quad (19)$$

Using a Flory-type [7] argument known from polymer physics and the theory of self-avoiding walks [8], de Gennes [9] estimated the fractal dimension for uncorrelated site percolation to be

$$D_{\text{dG}} = (d+2)/2. \quad (20)$$

In Fig. 1, the Fisher exponent τ with this estimate is compared with numerical ($d=3,4,5$) and analytic ($d=2,6$) results taken from Ref. [6].

Considering Eq. (4) at criticality with a long-distance cutoff $x_{\text{max}} = L$, corresponding to $s_{\text{max}} = L^D$, we obtain a relation between the fractal dimension and the critical exponent η_{per} characterizing the algebraic behavior (5) of the correlation function,

$$D = \frac{1}{2}(d+2 - \eta_{\text{per}}). \quad (21)$$

The de Gennes's estimate (20) is equivalent to setting η_{per} to zero here.

To determine the upper critical dimension, it is prudent to consider uncorrelated percolation on a Bethe lattice, which is exactly solvable. Because of the absence of closed paths on such a lattice, it mimics an ordinary lattice with high dimen-

sionality, where one expects the same critical behavior as in the upper critical dimension. The cluster distribution on a Bethe lattice is given by Eq. (1) with $\tau = \frac{5}{2}$ and $\sigma = \frac{1}{2}$, corresponding to the critical exponents

$$\alpha = -1, \quad \beta_{\text{per}} = 1, \quad \gamma_{\text{per}} = 1, \quad \nu = \frac{1}{2}, \quad \eta_{\text{per}} = 0, \quad (22)$$

and the fractal dimension $D = 4$. These values are consistent with the scaling laws in $d = 6$, which can thus be identified as the upper critical dimension.

To summarize, percolation near criticality is specified by the two exponents σ and τ parametrizing the cluster distribution, both of which are related to the fractal dimension D via Eqs. (18) and (19). Given the value of τ and σ , the critical exponents can be obtained using scaling relations. Specifically,

$$\alpha = 2 - \frac{\tau - 1}{\sigma}, \quad \beta_{\text{per}} = \frac{\tau - 2}{\sigma}, \quad \gamma_{\text{per}} = \frac{3 - \tau}{\sigma},$$

$$\eta_{\text{per}} = 2 + d \frac{\tau - 3}{\tau - 1}, \quad \nu = \frac{\tau - 1}{d\sigma}, \quad D = \frac{d}{\tau - 1}. \quad (23)$$

III. IDEAL BOSE GASES

In this section, we wish to point out a close analogy between percolation and Bose-Einstein condensation in an ideal Bose gas. The partition function Z , describing this system in d space dimensions, can be written as an integral over momentum as [10]

$$\ln(Z) = -V \int \frac{d^d k}{(2\pi)^d} \ln(1 - e^{-E(k)/k_B T}), \quad (24)$$

ignoring the $k = 0$ contribution. In Eq. (24), V is the volume of the system, while $E(k) = k^2/2m - \mu$ is the single-particle spectrum, with m the mass of the particles and μ the chemical potential that accounts for a finite particle number density. In deriving this, the time coordinate was analytically continued to the imaginary axis, where it becomes a cyclic variable taking values only in the interval $[0, \hbar/k_B T]$, with k_B Boltzmann's constant. After a Taylor expansion of the logarithm, the partition function takes the familiar form of a fugacity $[= \exp(\mu/k_B T)]$ series [10]

$$\ln(Z) = \frac{V}{\lambda^d} \sum_w l_w, \quad (25)$$

where

$$l_w(T) = w^{-\tau} \exp(\mu w/k_B T), \quad (26)$$

with $\tau = d/2 + 1$ and $\lambda = \hbar \sqrt{2\pi/mk_B T}$ the de Broglie thermal wavelength.

The correlation function $G(x)$ written as a sum over w reads

$$G(x) := \sum_w G_w(x) = \frac{1}{\lambda^d} \sum_w w l_w \exp\left(-\frac{\pi x^2}{\lambda^2 w}\right), \quad (27)$$

so that

$$G(0) = \frac{1}{\lambda^d} \sum_w w l_w \quad (28)$$

and

$$\int d^d x G(x) = \sum_w e^{\mu w/k_B T}, \quad (29)$$

where the left side of the last equation denotes $\tilde{G}(0)$, with $\tilde{G}(k)$ indicating the Fourier transform of $G(x)$. Physically, the correlation function $G_w(x)$ denotes the probability for a Brownian random walk starting at the origin to end up at a distance x from the origin after w steps, while $G_w(0)$ denotes the probability for the random walk to return to its starting point after w steps.

As last quantity, we need the correlation length ξ that is given by $\xi^2 = \lambda^2 k_B T / (-4\pi\mu)$, i.e., $\xi \propto \mu^{-1/2}$, as also follows from definition (16). The chemical potential is negative in the normal state and vanishes on approaching the critical temperature T_c , which is determined by the condition

$$n = G(0)|_{\mu=0}, \quad (30)$$

where n is the particle number density. This equation is valid for any $\mu < 0$ and thus for any temperature $T > T_c$. However, it starts to break down at the critical temperature where the condensate starts to form in the $k = 0$ state, which has been ignored in Eq. (24). The critical temperature is, in other words, the lowest temperature where this equation is still valid. From the criticality condition it follows that upon approaching the critical temperature from above, the chemical potential tends to zero in a d -dependent way as $\mu(T) \sim -(T - T_c)^{2/(d-2)}$. Below this temperature, in the condensed state, the chemical potential remains zero. The d dependence here distinguishes an ideal Bose gas from a Gaussian theory, where μ tends to zero as $\mu(T) \sim |T - T_c|$ irrespective of the dimensionality. The critical exponents of an ideal Bose gas, which can be extracted from the information given above, are $\beta = 1/2$, $\nu = 1/(d-2)$, and $\gamma = 2/(d-2)$. As noted a long time ago by Gunton and Buckingham [11], these exponents are the same as for the spherical model in d dimensions. This model corresponds to the limit $n \rightarrow \infty$ of the $O(n)$ spin model.

They continued to show that an ideal Bose gas with the modified energy spectrum $\epsilon(k) \propto k^D$, is in the same universality class as the spherical model with long-range interactions considered by Joyce [12]. The critical exponents for $D < d \leq 2D$, with $d = 2D$ the upper critical dimension, are given by

$$\alpha = \frac{d-2D}{d-D}, \quad \beta = \frac{1}{2}, \quad \gamma = \frac{D}{d-D},$$

$$\nu = \frac{1}{d-D}, \quad \eta = 2 - D. \quad (31)$$

The critical exponents of the standard ideal Bose gas are recovered by setting $D=2$. As will be shown shortly, the energy spectrum index D equals the fractal dimension.

The partition function of the ideal Bose gas with a modified energy spectrum can again be represented in the form (25), with $\tau=d/D+1$ and $\mu \propto (T-T_c)^{1/\sigma}$, where $\sigma=1/\nu D$ as for percolation [see Eqs. (18) and (19)]. The thermal wavelength λ for the generalized Bose gas is defined in a way that the spectrum can be written as

$$\epsilon/k_B T = c_D \lambda^D k^D, \quad (32)$$

with the constant c_D conveniently chosen such that

$$c_D^{d/D} = \frac{1}{(4\pi)^{d/2}} \frac{2}{D} \frac{\Gamma(d/D)}{\Gamma(d/2)}, \quad (33)$$

where $\Gamma(x)$ is the Euler gamma function. The correlation function $G(x)$ at $x=0$ is again given by Eq. (28) while also the integral of this function over space yields the same result as for $D=2$ given in Eq. (29). From the criticality condition (30), it follows that the correlation length scales with the chemical potential as $\xi \propto \mu^{-1/D}$, where—as already indicated by our choice of notation— D denotes the fractal dimension as calculated from its definition (15). The critical exponents (31) now follow easily.

As detailed in Ref. [13], the fugacity series (25) represents a sum over loops with the distribution (26), or, equivalently,

$$l_w(T) = (\ell/a)^{-\tau} \exp(-\theta \ell/k_B T), \quad \theta \propto (T-T_c)^{1/\sigma}, \quad (34)$$

with $l = wa$ the “length” measured in units of the characteristic “length” scale $a = \hbar k_B t$, and $\theta = -\mu k_B T / \hbar$ the string “tension.” The first factor in Eq. (34) measures the configurational entropy of loops, while the second is a Boltzmann factor. Each loop is characterized by the winding number w , telling how often it wraps around the imaginary time axis. Physically, a loop represents the worldlines in imaginary time of w particles grouped together in a single ring known

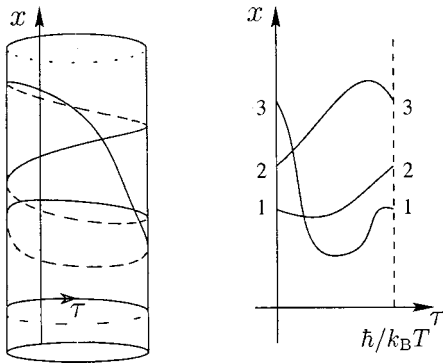


FIG. 2. A worldline loop involving three particles projected onto the finite imaginary time interval $0 \leq \tau \leq \hbar/k_B T$ (left panel). The loop, which has winding number $w=3$, can be equivalently represented as the worldlines of three cyclically permuted particles (right panel). (After Ref. [13].)

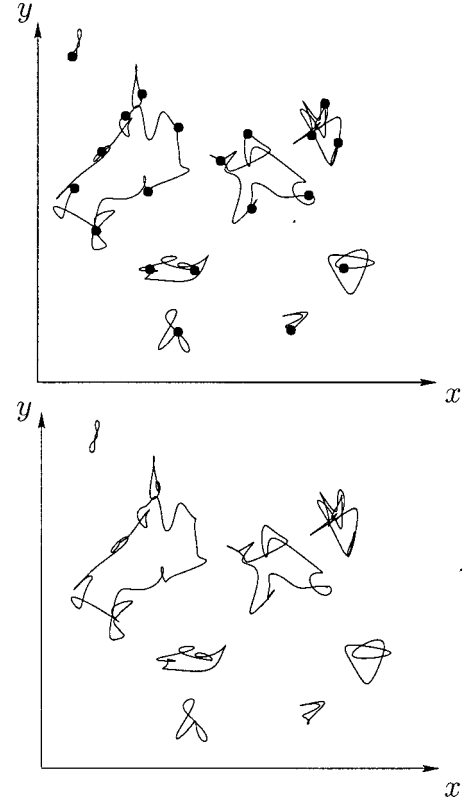


FIG. 3. Worldline loops of 20 particles in two space dimensions. Some of the worldlines are grouped together in single rings. The beads in the top panel represent particles moving in imaginary time along the strings towards their own initial position, in case of a single particle in a ring, or towards that of the neighboring particle in front of them. After an imaginary time $\hbar/k_B T$, these positions are reached and the particles cyclically permuted. In the bottom panel the beads are omitted, leaving behind a loop gas of worldlines. (After Ref. [13].)

from Feynman’s theory of the λ transition in ^4He [14]. Particles in the same ring are cyclically permuted after an imaginary time $\hbar/k_B T$ (see Fig. 2 for an illustration involving three particles). With increasing imaginary time, the particles move along closed strings towards the initial position of the particle in front of them (see Fig. 3). When only one particle is contained in a ring, the particle returns to its own initial position after an imaginary time $\hbar/k_B T$. On approaching the critical temperature from above, the chemical potential and thus the string tension become smaller, and loops with larger winding numbers start to appear. At T_c , the string tension vanishes and long loops containing arbitrarily many cyclically permuted particles appear in the system, signalling the onset of Bose-Einstein condensation. Above the critical temperature, long loops are exponentially suppressed. Because the loops are worldlines embedded in spacetime and parameterized by Euclidean time, l and a have the dimension of time, not of length—hence the quotation marks used below Eq. (34).

From this and also from comparing the partition function (25) of an ideal Bose gas with the one for percolation, Eq. (9), we observe a close connection between the two phenom-

ena. The rings of cyclically permuted particles in a Bose gas correspond to the clusters of percolation theory. The number of Bose particles contained in a ring corresponds to the number of sites contained in a cluster. In the same way that clusters grow when the percolation threshold is approached from below, rings with larger winding numbers appear in a Bose gas on approaching the critical point from above. In both cases, the behavior at the critical point is algebraic, rather than exponential. The percolating cluster at $p > p_c$ corresponds to the Bose-Einstein condensate at $T < T_c$. [The reason for ignoring the $k=0$ contribution in Eq. (24) was to bring out this analogy as clearly as possible.] But there are also differences.

Although the exponents (31) satisfy the scaling relations, β , γ , and η are not related to the exponents σ and τ parametrizing the loop distribution (26) in the same way as in percolation theory [see Eq. (23)]. Using these equations with the ideal Bose gas values for τ and σ , we find instead

$$\beta_{\text{per}} = 1, \quad \gamma_{\text{per}} = -\frac{d-2D}{d-D}, \quad \eta_{\text{per}} = 2 + d - 2D, \quad (35)$$

where we have given the exponents also the subscript ‘‘per’’ to indicate that we used the percolation formulas for them.

Despite the differences in form, these exponents too satisfy the scaling relations $2\beta_{\text{per}} + \gamma_{\text{per}} = 2 - \alpha$ and $\gamma_{\text{per}} = \nu(2 - \eta_{\text{per}})$, with α and ν given in Eq. (31). Apparently, there are two different sets of critical exponents, both satisfying the scaling laws.

To understand this, let us compare the sum rules (4) and (29) satisfied by the correlation function of percolation and an ideal Bose gas, respectively. There is no obvious connection between the two, implying that the ‘‘susceptibility’’ and also the ‘‘magnetization’’ are defined differently in the two systems. In percolation theory, they were defined in terms of the partition function in Eq. (10). With c replaced by $-\mu/k_B T$ to map the cluster distribution of percolation theory onto the loop distribution of an ideal Bose gas, it follows that P and S correspond to the condensate density n_0 and the compressibility κ of an ideal Bose gas, respectively, with

$$n_0 \sim (T_c - T)^{\beta_{\text{per}}}, \quad \kappa \sim |T_c - T|^{-\gamma_{\text{per}}}. \quad (36)$$

The exponents β and γ featuring in Eq. (31) are, on the other hand, connected to the order parameter of an ideal Bose gas, which is the field ϕ_0 describing the condensate, with $|\phi_0|^2 = n_0$. This last equation relates the two β exponents, giving $\beta_{\text{per}} = 2\beta$. Since $\beta_{\text{per}} = 1$, it also follows that $\gamma_{\text{per}} = -\alpha$, as one would expect naively. It should be noted that the exponents ν and α are not sensitive to the nature of the order parameter describing the transition. Given the presence of a single diverging length scale, any sensible definition of the correlation length leads to the same exponent ν . And α merely characterizes the algebraic behavior of the logarithm of the partition function, which is essentially the pressure. Despite the difference in the correlation function of percolation and an ideal Bose gas, the relation (18) between the correlation length exponent ν and the fractal dimension is the

same for the two, as can be easily checked repeating the derivation of this relation with the correlation function of an ideal Bose gas.

A more fundamental difference between percolation theory and the string picture of an ideal Bose gas is that in the latter, the number of particles is fixed, while in the former there is no condition on the number of occupied sites. The fixed number of particles in an ideal Bose led to the criticality condition (30) with $G(0)$ given in Eq. (28) that in turn determined the critical exponent ν . In percolation theory, where there is no such condition, ν is not determined that way. As a result, although the two phenomena are described by essentially the same theory specified by a loop or cluster distribution, they in general are not in the same universality class. Only the special case of percolation in the upper critical dimension $d_u = 6$, where $D = 4$, or on a Bethe lattice is in the same universality class as an ideal six-dimensional Bose gas with fractal dimension $D = 4$.

However, in the absence of a criticality condition, we expect uncorrelated percolation and strings of the type appearing in the description of an ideal Bose gas to be described by exactly the same theory. This is indeed what has been observed in numerical simulations on random string networks [15].

IV. RANDOM STRING NETWORKS

These simulations describe theories with a global $U(1)$ gauge symmetry, which undergo a phase transition from a symmetric state to one where the $U(1)$ symmetry is spontaneously broken. The ordered state, which is characterized by a complex order parameter, can have topological line defects—known as cosmic strings in the context of cosmology, and as vortices in the context of condensed matter. The line defects are either closed or terminate at the boundary; they cannot terminate inside the system. Upon circling a vortex of unit strength, the phase ϑ of the order parameter changes continuously by 2π .

To numerically study line defects, Vachaspati and Vilenkin [16] considered a cubic lattice with periodic boundary conditions. They discretized the vacuum manifold of the ordered state by allowing ϑ to take on only certain discrete values in the interval $0 \leq \vartheta < 2\pi$. The restriction to this interval assures that only vortices of unit strength are generated. Each lattice site is assigned one of the discrete values at random. Phases at different lattice sites are therefore uncorrelated. Vortices are traced by going around each plaquette of the lattice once. If a phase difference of 2π is found upon returning to the site one started at, it is concluded that a vortex penetrates the plaquette. In going from one lattice site to a neighboring one, the geodesic rule introduced by Kibble [17] is implemented, assuming that the (discrete) changes in ϑ are minimal. When two instead of just one vortex are found to penetrate a unit cell, the two incoming and two outgoing vortex segments are randomly connected. There are different ways to do this, the simplest one being to choose equal probabilities as was done in Ref. [16]. But there are other choices, such as favoring those possibilities that generate longer loops [18]. In this way, a whole random network

of vortices (of unit strength) can be traced out.

Such a random string network is of relevance to the formation of topological defects after a rapid phase transition involving the spontaneous breakdown of some symmetry [19]. As was first proposed by Kibble [17] in the context of cosmology, causally disconnected pieces of the early universe may after a rapid temperature quench end up in different, but degenerate states of the broken-symmetry vacuum, characterized by different values for the phase ϑ of the order parameter. When these regions come together, the resulting frustration at the boundaries may lead to the formation of topological defects. Zurek [20] noted that a similar mechanism might operate in the context of condensed-matter physics and developed a qualitative theory, based on scaling arguments, of the formation of topological defects in time-dependent phase transitions. To test this scenario, superfluid $^3\text{He-B}$ in rotation was radiated with neutrons [21]. After a neutron-absorption event, in which a small region is briefly heated up to the normal phase, a vortex network was produced. Under the action of the Magnus force, some of the vortex loops in the network expand and connect to the container wall. They are then, still under the action of the Magnus force, pulled straight, and finally, end up in the center of the cylindrical container aligned parallel to the rotation axis, where they can be detected by means of NMR. The number of vortices produced in the neutron irradiation experiments was in agreement with Zurek's predictions. The vortex network was numerically simulated [22] with the Vachaspati-Vilenkin algorithm discussed above.

Besides finite vortex loops, also infinite vortices appear in a random network with high enough vortex densities. The division between finite vortex loops and infinite ones is of course somewhat arbitrary on a finite lattice. A possible choice is to classify vortices with a length larger than $2L^2$, where L is the lattice size, as infinite. Universal quantities, such as the critical exponents should be independent of the precise choice. The fractal dimension found in the initial simulations [16] were consistent with that of a Brownian random walk, $D=2$. This result can be understood [4] by realizing that a dense random network was simulated, where a single vortex experiences an effective repulsion from the neighboring vortex segments as the volume occupied by these segments is not available anymore. The situation is similar to a polymer in a dense solution, which also displays the structure of a Brownian random walk.

Without any external field, all phase values of the order parameter have equal probability. Vachaspati [2] lifted this degeneracy of the vacuum manifold by giving some values a bias, thus obstructing the formation of vortices and decreasing the vortex density. When studying the statistics of the random network as function of the bias, a threshold was found, above which infinite vortices are absent and finite loops exponentially suppressed. The sudden appearance of infinite vortices as the threshold is approached from the low-density side, was interpreted as a percolation process.

Strobl and Hindmarsh [15] carried out more extensive numerical simulations on the transition of a random string network triggered by changing the bias. Using the percolation set of critical exponents, they measured—within statistical

errors—values similar to those for three-dimensional uncorrelated site percolation, which has $\sigma=0.45$, $\tau=2.18$ and $D=2.53$ [6]. For the vortex loops, the value $D>2$ means that they are self-seeking. The conclusion that the phase transition of a random string network is identical to the uncorrelated percolation transition is precisely what one would expect given our observations of the last section.

V. CORRELATED PERCOLATION

Up to this point we considered only *uncorrelated* site percolation, where lattice sites were occupied at random, and nearest-neighboring occupied sites grouped together in clusters. However, as is known from work on the Ising model [6], this is not the whole story.

In the context of the Ising model, it is more appropriate to refer to occupied sites as sites with spin up, say. In general, the percolation threshold temperature is below the critical one. That is to say, already before the thermal phase transition is reached, clusters of spin-up sites percolate the lattice. Only on a two-dimensional square lattice, the site percolation threshold temperature coincides with the critical temperature. Despite this, the Ising model and percolation critical exponents are different—as in higher dimensions. The situation is similar to the one discussed at the end of Sec. III, with the percolation transition being in a different universality class as the thermal one.

It is nevertheless possible to modify the definition of clusters in such a way that the critical behavior of the modified clusters becomes identical to that of the Ising model [23]. This approach goes back to seminal work by Fortuin and Kasteleyn [24], who mapped the partition function of the Ising model onto a *correlated* percolation problem. To obtain a modified cluster, take two nearest-neighboring sites i,j of a standard cluster with all the spins up, and add the pair to the new cluster with the bond probability $p_{ij}=1-\exp(-2J/k_B T)$, where J is the spin-spin coupling of the Ising model. The factor $2J$ appearing here corresponds to the increase in energy when one of the two spins involved in the bond is flipped. The upshot is that the bond clusters are in general smaller than the standard ones and also more loosely connected. The bond clusters percolate right at the critical temperature and display critical behavior identical to that of the Ising model.

The method has been turned into a powerful Monte Carlo algorithm by Swendsen and Wang [25], and by Wolff [26], where no individual spins are updated, but entire clusters. The main advantage of the nonlocal cluster algorithm in comparison to a local update algorithm, like Metropolis or heat bath, is that it drastically reduces the critical slowing down near the critical point.

Very recently, also the critical behavior of the three-dimensional $O(n)$ spin models, with $n=2,3,4$, was described as a correlated percolation process [27]. The relevant clusters percolate at the critical temperature, and were shown to have the same critical exponents as the corresponding models.

VI. WORLDLINE LOOP GASES

In the same way that correlated percolation can describe thermal phase transitions of interacting systems also corre-

lated worldline loop gases can. Recall that the loop gas description of an ideal Bose gas hinged on the representation (25) of the partition function in terms of the loop distribution (34), which derived from the representation (24) as an integral over momentum. If we assume that near the critical point, the partition function of an interacting system can be put in a similar form, but with $E(k)$ replaced by the single-particle spectrum of the excitation that becomes critical at T_c , the integral can in principle be done, yielding some loop distribution l_w , with w physically denoting the number of times a world-line loop wraps around the imaginary time axis. As for the ideal Bose gas, this distribution is parametrized by two exponents σ and τ , and can be put in the general form (34), with values characteristic for the universality class under consideration. The string tension of the worldline loops vanishes at T_c , and loops with arbitrarily large winding number w appear in the system, signalling the onset of Bose-Einstein condensation in the low-temperature phase. Note that the representation (24) of the partition function as an integral over momentum does not require a non-relativistic theory. Also relativistic theories lead to a similar form for the (equilibrium) partition function at finite temperature [28], as it is essentially fixed by Bose statistics.

The generalization of the sum rule (29) to correlated worldline loop gases reads

$$\int d^d x G(x) = \sum_w e^{-\theta/k_B T}, \quad (37)$$

where we recall that the string tension behaves near the critical point as $\theta \propto (T - T_c)^{1/\sigma}$. The sum rule leads directly to the relation

$$\gamma = 1/\sigma, \quad (38)$$

while at criticality, with the long distance cutoff $x_{\max} = L$, corresponding to $w_{\max} = L^D$ as before, it gives the relation

$$\eta = 2 - D. \quad (39)$$

These two relations generalize the corresponding results in Eq. (31) for an ideal Bose gas with modified energy spectrum. The value $\eta = 0$ separates self-avoiding worldlines ($D < 2$) from self-seeking ones. Since the fractal dimension has to be positive and smaller than the dimension d of the embedding space [29], $2 - d < \eta < 2$. Translated in terms of limits on τ , it follows that $2 < \tau < d/2 + 1$ for self-seeking worldlines, while $\tau > d/2 + 1$ for self-avoiding ones. Note that in two space dimensions and below, no self-seeking strings are possible.

As in Sec. III the exponents here are those for describing the thermal-phase transition of a loop gas of strings (worldlines). They are to be distinguished from the ones defined in the context of percolation theory given in Eq. (23). The two sets have, however, the exponents α and ν in common, implying that the present set can also be expressed entirely in terms of the exponents σ and τ parametrizing the loop distribution (34). Indeed, we find

TABLE I. Critical exponents for Brownian and smooth random walks together with the exponents σ and τ parametrizing the loop distribution (34).

Random walk	σ	τ	α	β	γ	η	ν	D
Brownian	1	$1 + d/2$	$2 - d/2$	$d/4 - 1/2$	1	0	$1/2$	2
Smooth	1	$1 + d$	$2 - d$	$d/2 - 1/2$	1	1	1	1

$$\alpha = 2 - \frac{\tau - 1}{\sigma}, \quad \beta = \frac{\tau - 2}{2\sigma}, \quad \gamma = \frac{1}{\sigma}, \quad \eta = 2 - \frac{d}{\tau - 1},$$

$$\nu = \frac{\tau - 1}{d\sigma}, \quad D = \frac{d}{\tau - 1}. \quad (40)$$

Since only general scaling laws are used in deriving these results, we believe they hold for any critical theory specified by a loop distribution (34) with exponents σ and τ . It can be easily verified that for an ideal Bose gas with $\sigma = d/D - 1$ and $\tau = d/D + 1$, Eq. (40) reduces to the correct values [Eq. (31)].

As an illustration, let us discuss the worldline loop distribution of some well-known statistical models. We start with random walks. In Table I the critical exponents of two different types of random walks are given together with the corresponding values for σ and τ parametrizing the loop distribution (34). Both have the value $\sigma = 1$ in common. The first one is the Brownian random walk [7], which has a fractal dimension $D = 2$, while the second one is the so-called smooth random walk [30,31], which has $D = 1$ and is therefore self-avoiding. Whereas a Brownian random walk can be understood as a particle hopping through a lattice with its next (nearest-neighboring) site chosen randomly at each step, a smooth random walk can be understood as a particle hopping through the lattice with its velocity rather than its position being changed randomly at each step. As a result, whereas a typical path of a Brownian random walk is continuous but not differentiable, a typical path of a smooth random walk is differentiable while only the first derivative is not. The reluctance to change directions is typical for fermionic particles. To understand the value $\eta = 1$ for a smooth random walk, note that the correlation function of a Brownian random walk is given in the continuum by

$$G(x) = \int \frac{d^d k}{(2\pi)^d} G_B(k) e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (41)$$

with

$$G_B(k) = \frac{1}{k^2 + \xi^{-2}}, \quad (42)$$

leading to the algebraic behavior (5) at criticality with $\eta = 0$. A smooth random walk, on the other hand, has a correlation function involving one factor of momentum less [30,31],

$$G_S(k) = \frac{1}{\mathbf{k} \cdot \mathbf{e} + \xi^{-1}}, \quad (43)$$

TABLE II. Critical exponents for the two-dimensional $O(n)$ spin model, with $n = -2, 0, 1, 2, \infty$, respectively, together with the exponents σ and τ parametrizing the loop distribution (34).

Model	n	σ	τ	α	β	γ	η	ν	D
Gaussian	-2	1	2	1	0	1	0	1/2	2
SAW	0	32/43	91/43	1/2	5/64	43/32	5/24	3/4	43/24
Ising	1	4/7	15/7	0	1/8	7/4	1/4	1	7/4
XY	2	0	15/7				1/4		7/4
Spherical	∞	0	2		1/2		0		2

where \mathbf{e} is some unit vector. At criticality, this again gives the algebraic behavior (5), but now with $\eta = 1$.

In two space dimensions, the critical exponents of various models are known exactly. For example, for $O(n)$ spin models with $n = -2 \cos(2\pi/t)$ and $1 \leq t \leq 2$ they are given by [32]

$$\alpha = \frac{2t-3}{t-2}, \quad \beta = \frac{(t-1)(t-3)}{8t(t-2)}, \quad \gamma = -\frac{t^2+3}{4t(t-2)},$$

$$\nu = -\frac{1}{2(t-2)}, \quad \eta = -\frac{(t-1)(t-3)}{2t}, \quad D = \frac{t^2+3}{2t}, \quad (44)$$

corresponding to

$$\sigma = -\frac{4t(t-2)}{t^2+3}, \quad \tau = \frac{(t+1)(t+3)}{t^2+3}. \quad (45)$$

For special values of the parameter t such that n is an integer, one obtains the critical exponents for the Gaussian model or the Brownian random walk ($n = -2$), the self-avoiding walk (SAW) ($n = 0$), the Ising model ($n = 1$), and the XY model ($n = 2$). The critical exponents for these models together with those of the spherical model are collected in Table II.

A few points may be worth noting.

(i) Both the Ising ($t = \frac{3}{2}$) and the XY model ($t = 2$) have the same fractal dimension. This is made possible by a minimum in $D(t)$ at $t = \sqrt{3}$ located between the two models, where $D = \sqrt{3}$.

(ii) In the literature, the fractal dimension is often equated to $1/\nu$ rather than to $1/\sigma\nu$ as in Eq. (18). We believe, the result (18) is more general.

(iii) The special character of the two-dimensional XY and also of the spherical model is seen in the loop gas picture by the vanishing of the exponent σ , implying the absence of the Boltzmann factor in the loop distribution function (34).

(iv) Besides the two-dimensional Ising and XY model there is at least one other one-dimensional object embedded in two space dimensions with the same fractal dimension. Surprisingly, it is the perimeter of the percolation cluster of standard, uncorrelated percolation in two dimensions [33], which can also be mapped onto various types of random walks. Close to criticality, the perimeter of the percolating cluster is again distributed according to Eq. (1) but with different values for the exponents. To avoid confusing with the two-dimensional cluster exponents $\sigma = 36/91$ and $\tau = 187/91$ corresponding to $D = 91/48$, we call the perimeter

exponents σ' and τ' . Since there is only one correlation length in the system, the cluster and perimeter exponents satisfy [34]

$$(\tau-1)/\sigma = (\tau'-1)/\sigma', \quad \sigma/\sigma' = D'/D. \quad (46)$$

The perimeter values [35]

$$\sigma' = 3/7, \quad \tau' = 15/7, \quad (47)$$

leading to $D' = 7/4$ indeed fulfill the relations (46).

Recent high-precision Monte Carlo studies of the three-dimensional XY universality class gave [36]

$$\nu = 0.6423(3), \quad \eta = 0.0381(2). \quad (48)$$

The resulting exponents parametrizing the worldline loop distribution are

$$\sigma = 0.7582, \quad \tau = 2.5291, \quad (49)$$

corresponding to a fractal dimension $D = 1.9619$ very close to 2, in accordance with the smallness of η . Since $D < 2$, the worldline loops of the three-dimensional XY model are self-avoiding.

VII. VORTEX LOOP GASES

Let us next turn to interacting systems with a global $U(1)$ gauge symmetry, which have vortices as topological defects in the ordered state. Thermal phase transitions in systems of this type have, besides the conventional Landau description, an alternative description in terms of proliferating vortices [37]. Such a scenario was first put forward by Onsager [1] and later by Feynman [38] to describe the λ transition in ^4He . In the ordered, superfluid state, where the $U(1)$ symmetry is spontaneously broken, only finite vortex loops are present. Because of the finite string tension, the loops are exponentially suppressed. Upon approaching the critical temperature from below, the string tension vanishes, and vortices start to proliferate. For vanishing string tension, vortices can gain configurational entropy by growing without energy cost. The resulting infinite vortices disorder the system and thereby restore the spontaneously broken $U(1)$ symmetry. The importance of vortex loop excitations in triggering the phase transition has since then been emphasized by various authors [39]. Early numerical evidence for this picture based on Monte Carlo simulations of the three-dimensional XY model was given by Janke and Kleinert [37,40]. Analytic

methods to describe the transition using vortex loops in a qualitative way were developed by Williams [41], and by Shenoy and collaborators [42].

Like worldline loops, vortex loops are specified by a loop distribution as in Eq. (34) parametrized by two exponents σ' and τ' [43]. These determine the critical exponents of the system again through Eq. (40), with σ and τ replaced by σ' and τ' . The result that the exponent $1/\sigma'$ parametrizing the vanishing of the vortex string tension directly gives the critical exponent γ —and not 2ν as is often claimed in the literature, also by the present author [13,44]—was first observed by Nguyen and Sudbø [45], while the connection $\eta=2-D$ for vortex loops was very recently derived independently in Ref. [29].

In the past few years, various groups have continued to numerically investigate the three-dimensional XY model from the perspective of vortices, in particular their loop distribution [18,45,46]. Let us discuss the numerical values for the exponents σ' and τ' obtained in these studies from the present perspective, concentrating on insights not available before.

Irrespective of the description chosen, either with the help of worldline loops or vortex loops, the expression for the correlation length exponent is formally the same. Since also the numerical value of this critical exponent is independent of the description, the exponents σ and τ parametrizing the worldline loop distribution and σ' and τ' parametrizing the vortex loop distribution satisfy $(\tau-1)/\sigma=(\tau'-1)/\sigma'$ as in Eq. (46). They should both lead to the same numerical value for ν given in Eq. (48). Using for σ' and τ' the numerical values [47]

$$1/\sigma' = 1.45(5), \quad \tau' = 2.4(1) \quad (50)$$

reported by Nguyen and Sudbø [45], we find from Eq. (40) a result $\nu=0.68$ very close to the expected one [Eq. (48)]. Since $2 < \tau' < d/2 + 1$, vortex loops in the XY model are (slightly) self-seeking. The numerical values for σ' and τ' reported by the other groups give the results $\nu=0.62$ [46] and $\nu=0.60$ [18], which are within about 10% of the expected result.

It was noted by Williams [41] that the fractal dimension D extracted from the value $\tau'=2.23(4)$ reported in Ref. [46] was close to the estimate $D=(d+2)/2$ obtained from a Flory-type of argument [42]. This estimate is identical to de Gennes's one [Eq. (20)] for uncorrelated percolation that was also based on a Flory-type argument. It should be noted, however, that de Gennes's modification of the original Flory argument was motivated in part to arrive at the correct upper critical dimension $d_u=6$ for percolation rather than $d_u=4$ for random walks and the XY model.

Kajantie and collaborators [18] investigated different algorithms to construct the string network. Depending on the algorithm chosen, they obtained (slightly) differing quantitative results. None of the algorithms used yielded a percolation threshold precisely at the critical temperature. From this observation they concluded that “geometrically defined percolation observable need not display universal properties related to the critical behavior of the system.” It has been suggested [48] that this discrepancy might instead hint at the

existence of a unique, yet to this date unknown, algorithm to construct the string network that would give precisely the expected XY critical behavior. The situation would then be similar to that found for correlated percolation and thermal phase transitions in $O(n)$ spin models discussed in Sec. V.

VIII. CONCLUSIONS

In this paper, we discussed the close analogy between percolation of (correlated) clusters and proliferation of strings at thermal-phase transitions of interacting systems. Two types of strings were considered: worldlines and topological line defects. Both are described in a way similar to clusters in percolation theory, being specified close to the critical point by a distribution parametrized by only two exponents. At the critical temperature, the string tension vanishes and loops proliferate in the same way as clusters percolate. If the loops are finite-temperature worldlines, their proliferation signals the formation of a Bose-Einstein condensate that spontaneously breaks the global symmetry characteristic of the phase transition under consideration. If, on the other hand, the loops are vortices, their proliferation signals the disordering of the ordered state in which these vortices are topologically stable.

In deriving our results, we used the ideal Bose gas with modified energy spectrum as a stepping stone. The model, despite being noninteracting, has nontrivial critical exponents that are known exactly. It also has the virtue that it can be mapped onto a loop gas in an exact way. The resulting relations between the exponents parametrizing the loop distribution and the critical exponents describing the thermal-phase transition of the system, were shown to be easily generalized to interacting loop gases using general scaling laws. As for percolation, the two exponents parametrizing the distribution were shown to encode the entire set of critical exponents.

Various statistical models with exactly known critical exponents were discussed from the perspective of a worldline loop gas. Also recent numerical studies on the statistical properties of the vortex loop gas in the three-dimensional XY model were discussed.

It was shown that a random string network is to thermal-phase transitions involving correlated strings as what uncorrelated site percolation is to thermal phase transitions in $O(n)$ spin models.

A possible connection of the approach discussed in this paper with the Monte Carlo loop algorithm introduced by Everetz [49] is presently under investigation.

Note added in proof. Recently I received a copy of Ref. [51] where the connection between Bose-Einstein condensation in a Bose gas and percolation was considered previously. I kindly thank the author for sending me the reprint.

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